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Existence of Limit Cycles in a Predator-Prey System With a Functional Response of the Form Arctan(ax)

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Abstract

We consider a predator prey system with the functional response of the form $\theta(x) = \arctan(ax)$; $a > 0$. The main concern in this paper is the existence of limit cycles for such system. A necessary and sufficient condition for the nonexistence of limit cycles is given for such system.

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1. Introduction

The study of some conditions under which a predator prey system has no limit cycles got the attention of many authors, see [4], [5], [7], [15]. They gave some criteria and conditions for the existence and nonexistence of such limit cycles for different functional response functions. For example Attili [3] and Kooij and Zegeling [10] have examined a predator prey system with Ivlev's response; that is, $\theta(x) = 1 - e^{-ax}$, $a > 0$. Holling type I predator-prey model was considered by Liu, Zhang and Chen [13]. Sugie [16] presented conditions under which systems with Ivlev's response have a unique limit cycle. Existence and uniqueness of limit cycles in general were studied by for example Hasik [6], Huang and Zhu [7] and Hwang [8]. Predator prey systems with a class of functional responses was considered by Hesaaraki and Moghadas [5], Hwang [8]

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and Liu and Yuan [12]. Global stability was considered by Kar [9] and for a general Gause-type model was considered by Ardito et al [2] where they constructed a Lyapunov function for the predator prey system to establish such stability. For the same model; that is, Gause-type, the question of uniqueness of limit cycles was answered by Kuang and Freedman [11] and the numerical computations were done by Moghadas, Alexander and Corbett [14]. More recently, Kar [9] and Ruan and Xiao [15] considered the global stability of predator prey systems with non-monotonic functional response. For numerical treatment of the problem we mention for example Arbogast and Milner [1] and Moghadas, Alexander and Corbett [14] who considered finite differences.

The study of existence of limit cycles has direct connection to the question of stability. If the predator prey system has a unique positive critical point, it is often predicted that there is equivalence between local and global stability of the critical point. One way to show the stability of the positive critical point is to show that the system has no limit cycle. This is the approach we follow.

We deal with a general predator prey model of the form

$$\frac{dx}{dt} = rx(1-x) - y\theta(x); \quad \frac{dy}{dt} = -Dy + sy\theta(x), \tag{1.1}$$

where x and y are the prey and the predator population sizes respectively, r, s and D are positive parameters while $\theta(x) = \arctan(ax)$; $a > 0$ satisfying

i) $\theta(0) = 0$ and $\theta'(x) > 0$ for $x \geq 0$ while $\theta''(x) < 0$ and $\theta'''(x) > 0$ for $x > 0$,
ii) $\lim_{x\to\infty} \theta(x)$ is finite.

(1.2)

The purpose of this work is to investigate the question of nonexistence of limit cycles for the system (1.1) subject to (1.2). In the next section we present necessary conditions for the absence of limit cycles while the sufficient condition is given in the last section.

2. Non-existence of Limit Cycles – Necessary Condition

We start this section by some results on the system (1.1). In particular, we are interested in its critical points. Note that for a suitable $a > 0$ the system (1.1) has a unique critical point in the first quadrant if

$$k\pi < \frac{D}{s} < k\pi + \frac{\pi}{2} \tag{2.1}$$

where k is an integer. That is, (x^*, y^*) is the critical point with

$$x^* = \frac{1}{a}\tan\frac{D}{s}; \quad \text{and} \quad y^* = \frac{rsx^*(1-x^*)}{D}. \tag{2.2}$$

Here D, s and a are chosen such that $0 < x^* < 1$. If (2.1) is not satisfied, then the system (1.1) will not have any critical points in the first quadrant. This means we can assume $a > 0$ and $0 < x^* < 1$. It is easily seen also that (0,0) and (1,0) are two saddle points for the system.

Assume (x^*, y^*) is a critical point in the first quadrant and consider the linearized system at (x^*, y^*) which has the matrix form

$$A = \begin{bmatrix} -(2rx^* + \dfrac{ay^*}{1+(ax^*)^2} - r) & -\arctan(ax^*) \\ \dfrac{say^*}{1+(ax^*)^2} & 0 \end{bmatrix}. \tag{2.3}$$

One can easily see that the characteristic polynomial of A has roots with positive real parts if and only if

$$2rx^* + \frac{ay^*}{1+(ax^*)^2} - r < 0 \tag{2.4}$$

since $\dfrac{say^* \arctan(ax^*)}{1+(ax^*)^2} > 0$. This leads to the following result

Theorem 2.1: If the system (1.1) has no limit cycles, then

$$2rx^* + \frac{ay^*}{1+(ax^*)^2} - r \geq 0.$$

Now looking at the result of Theorem 1 in more details and substituting x^* and y^* given by (2.2) will lead to

$$\frac{2r}{a}\tan\frac{D}{s} + \frac{rs\tan\frac{D}{s}\left[1 - \frac{1}{a}\tan\frac{D}{s}\right]}{D\left[1+\tan^2\frac{D}{s}\right]} - r \geq 0 \text{ or } \frac{2r}{a}\tan\frac{D}{s} + \frac{r\tan\frac{D}{s}\left[1 - \frac{1}{a}\tan\frac{D}{s}\right]}{\frac{D}{s}\left[1+\tan^2\frac{D}{s}\right]} \geq r.$$

Simplifying and combining terms on the left side will lead to

$$\tan\frac{D}{s}\left[\frac{2}{s} + \frac{a - \tan\frac{D}{s}}{D\left[1+\tan^2\frac{D}{s}\right]}\right] \geq a \text{ or } \tan\frac{D}{s}\left[\frac{2D\left[1+\tan^2\frac{D}{s}\right] + sa - s\tan\frac{D}{s}}{D\left[1+\tan^2\frac{D}{s}\right]}\right] \geq a.$$

This implies

$$\tan\frac{D}{s}\left[\frac{s\tan\frac{D}{s} - 2D\left[1+\tan^2\frac{D}{s}\right] - sa}{D\left[1+\tan^2\frac{D}{s}\right]}\right] \leq -a.$$

Multiplying by minus and collecting terms lead to

$$\tan\frac{D}{s}\left[s\tan\frac{D}{s} - 2D\left[1+\tan^2\frac{D}{s}\right] - sa\right] \geq aD\left[1+\tan^2\frac{D}{s}\right].$$

or

$$\tan\frac{D}{s}\left[s\tan\frac{D}{s} - 2D\left[1+\tan^2\frac{D}{s}\right]\right] \geq as\tan\frac{D}{s} - aD\left[1+\tan^2\frac{D}{s}\right].$$

Finally taking a as common factor on the right side and dividing by $s\tan\dfrac{D}{s} - D\left[1+\tan^2\dfrac{D}{s}\right]$ we obtain the formula

$$\tan\frac{D}{s}\left[\frac{s\tan\frac{D}{s} - 2D\left[1+\tan^2\frac{D}{s}\right]}{s\tan\frac{D}{s} - D\left[1+\tan^2\frac{D}{s}\right]}\right] \geq a. \tag{2.5}$$

As a result, the necessary condition for the nonexistence of limit cycles is given by (2.5).

3. Sufficient Condition for the Nonexistence of Limit Cycles

Consider the Lie'nard system of the form

$$\frac{du}{ds} = h(v) - f(u); \quad \frac{dv}{ds} = g(u) \tag{3.1}$$

where f, g and h are real valued continuous functions on $I = (-b, c)$ with $b, c > 0$ and can be infinite, see Attili [3]. Assume also that

$$f(0) = 0,\ h(0) = 0,\ \ g(0) = 0 \text{ and } ug(u) > 0 \text{ if } u \neq 0 \text{ and } vh(v) > 0 \text{ if } v \neq 0. \tag{3.2}$$

Now let

$$G(u) = w = \int_0^u |g(\xi)|\, d\xi, \tag{3.3}$$

since $G(u)$ is strictly increasing, this means $G^{-1}(u)$ exists for $|w| < M$ with $M = \min\{M^+, M^-\}$ where $M^+ = \int_0^{+\infty} |g(\xi)|\, d\xi$ and $M^- = \int_{-\infty}^0 |g(\xi)|\, d\xi$.

Theorem 3.1: Assume (3.2) holds and

$$f(G^{-1}(-w)) \neq f(G^{-1}(w)) \text{ for } 0 < w < M. \tag{3.4}$$

Then the system (1.1) has no limit cycles (periodic solutions) in the set $\{(u, v) : u \in I, v \in R\}$ except for the origin.

For the proof, see Theorem 3.8 in Sugie and Hara [16]. Now to make use of this result, we start by a change of variables

$$u = x - x^*,\ v = \log\frac{y}{y^*} \text{ and } ds = -\arctan(ax)dt,$$

then (1.1) is transformed into a Lie'nard system with

$$f(u) = r(u + x^*)\frac{1 - (u + x^*)}{\arctan(a(u + x^*))} - y^*;$$

$$g(u) = \frac{s \arctan(a(u + x^*)) - D}{\arctan(a(u + x^*))} \quad \text{and} \quad h(v) = y^*(e^v - 1).$$

It is clear that $f(0) = g(0) = h(0) = 0$ and for all $u \in I$ and $v \in R$, $\frac{dg}{du}(u) > 0$ and $\frac{dh}{dv}(v) > 0$. Hence (3.2) is satisfied. Now it remains to show that (3.4) is satisfied for this choice of f, g and h. Now we present some discussions and results that lead to such inequality.

Consider

$$f'(u) = \frac{\arctan(az)\,[1 - 2z]\,[1 + a^2z^2] - arz\,(1 - z)}{[1 + a^2z^2]\,(\arctan(az))^2},$$

with $z = u + x^*$. The denominator is clearly positive. Let us investigate the sign of the numerator; that is, $R(z) = \arctan(az)\,[1 - 2z]\left[1 + a^2z^2\right] - arz\,(1 - z)$. Differentiating $R(z)$ with respect to u leads to

$$R'(z) = \left[-6a^2z^2 + 2a^2z - 2\right]\arctan(az). \tag{3.5}$$

Now on $I = (-x^*, +\infty)$ and for $u > -x^*$ one concludes that $R'(z) \leq 0$ if $a^2 \leq 12$. This means $R(z)$ is decreasing for $u > -x^*$. Now since $R(-x^*) = 0$, then $f'(u) \leq 0$ for $u > -x^*$; that is, $f(u)$ is decreasing on I. As a result we have established the following Lemma.

Lemma 3.2: If $a^2 \leq 12$ then $f(u)$ is decreasing on $I = (-x^*, +\infty)$.

It can be easily seen that if $a^2 > 12$ then there exist u_1, u_2 with $u_1 < u_2$ such that $R'(u_1) = R'(u_2) = 0$ and $R'(u_1) > 0$ if $u \in (u_1, u_2)$; that is, $R(u)$ is increasing if $u \in (u_1, u_2)$ and $R'(u_1) < 0$ if $u \in I - [u_1, u_2]$ which means $R(u)$ is decreasing if $u \in I - [u_1, u_2]$. Otherwise there exist v_1, v_2 such that $-x^* < v_1 < v_2$ and $R(v_1) = R(v_2) = f'(v_1) = f'(v_2) = 0$, also $f'(u) > 0$ if $u \in (v_1, v_2)$ and $f'(u) < 0$ if $u \in I - [v_1, v_2]$. Therefore v_1 and v_2 has the same sign. This type of discussion leads to the following results.

Lemma 3.3: Assume (2.5) is satisfied. If $a^2 > 12$, then one of the following statements holds:

a) $f(u)$ is decreasing for $u > -x^*$, or

b) There exist v_1 and v_2 with same sign and $f'(u) > 0$ if $u \in (v_1, v_2)$ and $f'(u) < 0$ if $u \in I - [v_1, v_2]$.

Lemma 3.4: Assume (2.5) is satisfied. If $a^2 \leq 12$ or $a^2 > 12$ and $f(u)$ is decreasing on I, then the system (1.1) has no limit cycles.

Proof: It is sufficient to show that $f(G^{-1}(-\gamma_0)) \neq f(G^{-1}(\gamma_0))$ and then apply Theorem 1. For that reason, assume the contrary; that is, there is a $\gamma_0 > 0$ such that $f(G^{-1}(-\gamma_0)) = f(G^{-1}(\gamma_0))$. Then $f(-\alpha) = f(\beta)$ and $G(-\alpha) = \gamma_0 = G(\beta)$ where $\alpha = -G^{-1}(\gamma_0)$, $\beta = G^{-1}(\gamma_0)$ and $-x^* < -\alpha < 0 < \beta$. Since $f(0) = 0$, then from Lemmas 2 and 3, we have $f(u) \geq 0$ for $-x^* < u \leq 0$ and $f(u) < 0$ for $u > 0$. This leads to contradiction. This means $f(-\alpha) \neq f(\beta)$ for every $\alpha, \beta > 0$, completing the proof. ∎

With this we have established that (2.5) is sufficient condition for the nonexistence of limit cycles of (1.1) if part (a) of Lemma 3 is satisfied.

Now suppose that part (b) of Lemma 3 occurs. If we look back at $R(u) = [-6a^2z^2 + 2a^2z - 2]\arctan(az)$, again with $z = u_0 + x^*$, we see that $v_1 < v_2 \leq 0$ if $x^* \geq \frac{1}{6}$ and $0 < v_1 < v_2$ if $x^* < \frac{1}{6}$ where v_1 and v_2 are given in Lemma 3. Assume $u_0 > 0$ is a root of $f'(u) = 0$. Then applying L'Hopitals rule, we have

$$\frac{1}{a}\left[r(1-2z)(1+a^2z^2)\right] = \frac{rz(1-z)}{\arctan az}. \tag{3.6}$$

Substituting in $f(u)$, we get $f(u_0) = \dfrac{r(1-2z) - y^*\dfrac{a}{1+a^2z^2}}{\dfrac{a}{1+a^2z^2}}$. Again using a similar argument as before define the numerator as

$$T(u) = r(1-2z) - y^*\frac{a}{1+a^2z^2}, \tag{3.7}$$

differentiating with respect to u and since $z = u_0 + x^*$ we obtain

$$T'(u) = -2r - y^*\frac{-2a^3(u+x^*)}{\left[1+a^2(u+x^*)\right]^2}. \tag{3.8}$$

If $\bar{u} > 0$ such that $T'(\bar{u}) = 0$, this will imply that $ay^* = \dfrac{r\left[1+a^2(\bar{u}+x^*)\right]^2}{a^2(\bar{u}+x^*)}$. Substituting in (3.6) leads to

$$T(\bar{u}) = r\left[1 - 2(\bar{u}+x^*) - \frac{1+a^2(\bar{u}+x^*)^2}{a^2(\bar{u}+x^*)}\right] < 0. \tag{3.9}$$

If (1.1) holds, this means $T(0) \leq 0$ and hence from (3.8) $\lim_{\bar{u}\longrightarrow+\infty} T(\bar{u}) = -\infty$. Thus $T(u) < 0$ for $u > 0$ and as a result $f(u_0) < 0$. Notice that if $x^* < \frac{1}{6}$ then $0 < v_1 < v_2$

and $f'(v_1) = f'(v_2) = 0$,which means $f(v_1) < 0$ and $f(v_2) < 0$. With this we have the following result.

Lemma 3.5: If (2.5) holds then

a) If $a^2 > 12$, then $f(u) < 0$ for $u > 0$.

b) If $a^2 > 12$ and $x^* \leq \dfrac{1}{6}$, then the system (1.1) has no limit cycles.

The proof is clear by using part (b) of Lemma 3 and the argument before the lemma since they establish $f(u) < 0$ for $u > 0$ and $f(u) \geq 0$ for $u \leq 0$. Therefore the inequality (3.4) is satisfied.

For the case $x^* > \dfrac{1}{6}$ one can take advantage of Theorem 3.1 in Ardito and Ricciardi [2] to prove the following result which is complementary to Lemma 5.

Lemma 3.6: Assume (2.4) holds. If $a^2 > 12$ and $x^* > \dfrac{1}{6}$, then the system (1.1) has no limit cycles.

Now we can state the following theorem that establishes the necessary and sufficient condition for the nonexistence of limit cycles of (1.1).

Theorem 3.7: Assume that (x^*, y^*) is a critical point of (1.1). Then the system has no limit cycles if and only if

$$\tan\frac{D}{s}\left[\frac{s\tan\frac{D}{s} - 2D\left[1+\tan^2\frac{D}{s}\right]}{s\tan\frac{D}{s} - D\left[1+\tan^2\frac{D}{s}\right]}\right] \geq a.$$

Proof: The necessary condition was given in Section 2, Theorem 1. While the sufficient condition was given by Lemma 4 for the case $a^2 \geq 12$ or $a^2 > 12$ and $f(u)$ is decreasing and in Lemmas 5 and 6 for the other case when $a^2 > 12$ and $x^* \leq \dfrac{1}{6}$ or $x^* > \dfrac{1}{6}$. ■

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